

ULRICH BUNDLES ON BLOWUPS

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ABSTRACT. We construct an Ulrich bundle on the blowup at a point when the original variety is embedded by a sufficiently positive linear system and carries an Ulrich bundle. In particular, we describe the relation between special Ulrich bundles on the blown-up surfaces and the original surfaces.

Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n , embedded by a complete linear system $|\mathcal{O}_X(H)|$ for some very ample divisor H . An *Ulrich bundle* on X [8] is a vector bundle \mathcal{F} on X whose twists satisfy a set of vanishing conditions on cohomology

$$H^i(X, \mathcal{F}(-jH)) = 0 \text{ for all } i \text{ and } 1 \leq j \leq n.$$

Ulrich bundles appeared in commutative algebra in relation with maximally generated maximal Cohen-Macaulay modules [14]. In algebraic geometry, the notion of Ulrich bundles surprisingly appeared thanks to recent works by Beauville and Eisenbud-Schreyer. The importance is motivated by the relations between the Cayley-Chow forms [2, 8] and with the cohomology tables [9].

Eisenbud and Schreyer made a conjecture that every projective variety admits an Ulrich bundle [8], which is wildly open even for smooth surfaces. The answer is known for a few cases including: curves [8], complete intersections [11], Grassmannians [7], del Pezzo surfaces [8] and more rational surfaces with an anticanonical pencil [12], general K3 surfaces [1], abelian surfaces [3], Fano polarized Enriques surfaces [5], and surfaces with $q = p_g = 0$ embedded by a sufficiently large linear system [4].

In classical algebraic geometry, there are 2 fundamental operations, namely, the hyperplane cut and the linear projection. It is well known that the restriction of an Ulrich bundle to a general hyperplane section is also an Ulrich bundle (cf. [6]). It is also straightforward that the vanishing conditions do not affect on taking a linear projection from a point $P \in \mathbb{P}^N \setminus X$. Hence, the only interesting case occurs from the “projection” from a point inside of X which can be realized as the blowup at a point.

We briefly review the relation between inner projections and blowups. Let $P \in X$ be a point. The linear projection from P gives a rational map $\pi_P : X \dashrightarrow \mathbb{P}^{N-1}$ defined on $X \setminus \{P\}$. We can eliminate the point of indeterminacy by taking the blow-up $\sigma : \tilde{X} \rightarrow X$ at P . The complete linear system $|\sigma^*\mathcal{O}_X(H) \otimes \mathcal{O}_{\tilde{X}}(-E)|$ induces a morphism from \tilde{X} to \mathbb{P}^{N-1} whose image is the closure of $\pi_P(X \setminus \{P\})$, where $E = \sigma^{-1}(P)$ is the exceptional divisor.

In this short note, we construct an Ulrich bundle on the blowup at a point from an Ulrich bundle on the original variety.

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Theorem 1. *Assume furthermore that the divisor $\tilde{H} := \sigma^*H - E$ is very ample. Suppose we have an Ulrich bundle \mathcal{F} on X with respect to the polarization $\mathcal{O}_X(H)$. Then the vector bundle*

$$\tilde{\mathcal{F}} := \sigma^*\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(-E)$$

is an Ulrich vector bundle on \tilde{X} with respect to $\mathcal{O}_{\tilde{X}}(\tilde{H})$.

Proof. We have to show that $\tilde{\mathcal{F}}(-j\tilde{H}) = \sigma^*(\mathcal{F}(-jH)) \otimes \mathcal{O}_{\tilde{X}}((j-1)E)$ has no cohomology for every $1 \leq j \leq n$. Note that the push-forward $\sigma_*\mathcal{O}_{\tilde{X}}(jE) = \mathcal{O}_X$ for every $j \geq 0$. We first claim that the higher direct image $R^i\sigma_*\mathcal{O}_{\tilde{X}}((j-1)E) = 0$ for every $i > 0$ and $1 \leq j \leq n$. It is enough to show that the stalk vanishes at every point $Q \in X$, which can be computed from the inverse limit

$$(R^i\sigma_*\mathcal{O}_{\tilde{X}}((j-1)E))_Q^\wedge = \begin{cases} 0 & Q \neq P, \\ \varprojlim H^i(mE, \mathcal{O}_{mE}((j-1)E)) & Q = P. \end{cases}$$

By the short exact sequence

$$0 \rightarrow \mathcal{O}_E(-(m-1)E) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(m-1) \rightarrow \mathcal{O}_{mE} \rightarrow \mathcal{O}_{(m-1)E} \rightarrow 0,$$

we have

$$\begin{aligned} H^i(mE, \mathcal{O}_{mE}((j-1)E)) &\simeq H^i((m-1)E, \mathcal{O}_{(m-1)E}((j-1)E)) \\ &\vdots \\ &\simeq H^i(E, \mathcal{O}_E((j-1)E)) \\ &= H^i(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1-j)) = 0 \end{aligned}$$

for any $i > 0, m \geq 1$ and $1 \leq j \leq n$. Applying the projection formula, we have

$$\begin{aligned} R^i\sigma_*(\tilde{\mathcal{F}}(-j\tilde{H})) &= \mathcal{F}(-jH) \otimes R^i\sigma_*\mathcal{O}_{\tilde{X}}((j-1)E) \\ &= 0 \end{aligned}$$

for every $i > 0$ and $1 \leq j \leq n$. Hence, Leray spectral sequence implies that the cohomology group

$$\begin{aligned} H^i(\tilde{X}, \tilde{\mathcal{F}}(-j\tilde{H})) &\simeq H^i(X, \sigma_*(\tilde{\mathcal{F}}(-j\tilde{H}))) \\ &\simeq H^i(X, \mathcal{F}(-jH) \otimes \sigma_*\mathcal{O}_{\tilde{X}}((j-1)E)) \\ &= H^i(X, \mathcal{F}(-jH)) \\ &= 0 \end{aligned}$$

vanishes for every i and $1 \leq j \leq n$, since \mathcal{F} is Ulrich on (X, H) . Therefore, we conclude that $\tilde{\mathcal{F}}$ is an Ulrich vector bundle on (\tilde{X}, \tilde{H}) . \square

Particularly interesting case happens when X is a smooth surface. The above construction provides an Ulrich bundle on blown-up surfaces at a few points, by taking consecutive inner projections. Moreover, the procedure also provides a direct application on “special Ulrich bundles”. Eisenbud and Schreyer introduced the notion of *special Ulrich bundles* on a surface X [8], which are Ulrich bundles \mathcal{F} of rank 2 such that $\det \mathcal{F} = \mathcal{O}_X(K_X + 3H)$, where K_X denotes the canonical divisor of X . The existence of special Ulrich bundles yields a very nice presentation of the Cayley-Chow form of X , indeed, X admits a Pfaffian Bézout form in Plücker coordinates [8]. As an immediate consequence, the procedure with a special Ulrich bundle gives rise to a special Ulrich bundle on the upstairs:

Corollary 2. *Let (X, H) be a smooth polarized surface satisfies the assumptions in Theorem 1. If \mathcal{F} is a special Ulrich bundle on X , then $\tilde{\mathcal{F}}$ is also a special Ulrich bundle on \tilde{X} .*

Proof. It comes from a direct computation

$$\det \tilde{\mathcal{F}} = \sigma^*(\mathcal{O}_X(K_X + 3H)) \otimes \mathcal{O}_{\tilde{X}}(-2E) = \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + 3\tilde{H}).$$

□

It is also possible to construct a special Ulrich bundle in the converse direction, which reveals the connection between special Ulrich bundles on the upstairs and downstairs:

Theorem 3. *Let (X, H) be a smooth polarized surface satisfies the assumptions in Theorem 1 as above. Let $\tilde{\mathcal{F}}$ be a special Ulrich bundle on (\tilde{X}, \tilde{H}) . Then $\sigma_*(\tilde{\mathcal{F}}(E))$ is a special Ulrich bundle on (X, H) .*

Proof. We first claim that $\sigma_*(\tilde{\mathcal{F}}(E))$ is a vector bundle on X . Since $c_1(\tilde{\mathcal{F}}(E)) = K_{\tilde{X}} + 3\tilde{H} + 2E = \sigma^*(K_X + 3H)$, we have $\deg \tilde{\mathcal{F}}(E)|_E = 0$. By Grothendieck's theorem, we have $\tilde{\mathcal{F}}(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ for some $a \geq 0$. Note that $\tilde{\mathcal{F}}$ is globally generated since it is 0-regular with respect to \tilde{H} . Hence the restriction $\tilde{\mathcal{F}}|_E \simeq \mathcal{O}_{\mathbb{P}^1}(a+1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a+1)$ is also globally generated, so either $a = 0$ or $a = 1$ holds. For the both cases, we obtain $h^0(E, \tilde{\mathcal{F}}(E)|_E) = 2$. Therefore $h^0(\sigma^{-1}(Q), \tilde{\mathcal{F}}(E)|_{\sigma^{-1}(Q)}) = 2$ holds for every $Q \in X$, which implies that $\sigma_*(\tilde{\mathcal{F}}(E))$ is locally free of rank 2 by Grauert's theorem. Also note that a similar computation induces that $R^1\sigma_*(\tilde{\mathcal{F}}(E)) = 0$ since $\tilde{\mathcal{F}}(E)|_E$ has vanishing H^1 .

To prove $\sigma_*(\tilde{\mathcal{F}}(E))$ is a special Ulrich bundle, it is enough to show that $\det \sigma_*(\tilde{\mathcal{F}}(E)) \simeq \mathcal{O}_X(K_X + 3H)$ and partial vanishing conditions $H^\bullet(X, \sigma_*(\tilde{\mathcal{F}}(E)) \otimes \mathcal{O}_X(-H)) = 0$. Indeed, for those vector bundles, we have

$$\begin{aligned} H^i(X, \sigma_*(\tilde{\mathcal{F}}(E)) \otimes \mathcal{O}_X(-2H)) &= H^{2-i}(X, \sigma_*(\tilde{\mathcal{F}}(E))^* \otimes \mathcal{O}_X(K_X + 2H))^* \\ &= H^{2-i}(X, \sigma_*(\tilde{\mathcal{F}}(E)) \otimes \mathcal{O}_X(-H))^* \\ &= 0 \end{aligned}$$

by Serre duality.

Note that the determinant of a coherent sheaf \mathcal{G} is the alternating product $\bigotimes \det(\mathcal{E}_i)^{(-1)^i}$ where \mathcal{E}_i define

$$0 \rightarrow \mathcal{E}_r \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{G} \rightarrow 0$$

a finite locally free resolution of \mathcal{G} . It is well-known that such a locally free resolution always exists on a smooth variety, and the determinant is equal to the structure sheaf when \mathcal{G} is supported on a subset of codimension at least 2 (cf. [10]). Let $V \subset H^0(\tilde{X}, \tilde{\mathcal{F}})$ be a general subspace of dimension 3. Since $\tilde{\mathcal{F}}$ is globally generated, the evaluation map $ev : V \otimes \mathcal{O}_{\tilde{X}} \rightarrow \tilde{\mathcal{F}}$ is surjective possibly except for finitely many points. Hence we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}} - 3\tilde{H}) \simeq \sigma^*\mathcal{O}_X(-K_X - 3H) \otimes \mathcal{O}_{\tilde{X}}(2E) \rightarrow V \otimes \mathcal{O}_{\tilde{X}} \xrightarrow{ev} \tilde{\mathcal{F}} \rightarrow \mathcal{R}_Z \rightarrow 0.$$

Here, $\mathcal{R}_Z = (\ker ev)$ is supported on a finite set of points $Z \subset \tilde{X}$. Since \mathcal{R}_Z and $R^1\sigma_*$ terms have supports of codimension at least 2, they don't affect on

the determinant computation. Twisting by $\mathcal{O}_{\tilde{X}}(E)$ and taking push-forward, we conclude that

$$\begin{aligned} \det \sigma_*(\tilde{\mathcal{F}}(E)) &= \det(V \otimes \sigma_* \mathcal{O}_{\tilde{X}}(E)) \otimes (\det \sigma_*(\sigma^* \mathcal{O}_X(-K_X - 3H) \otimes \mathcal{O}_{\tilde{X}}(3E)))^* \\ &= (\wedge^3 V \otimes \mathcal{O}_X) \otimes \mathcal{O}_X(-K_X - 3H)^* \\ &= \mathcal{O}_X(K_X + 3H) \end{aligned}$$

as desired.

Apply the projection formula and Leray spectral sequence, we have

$$\begin{aligned} H^i(X, \sigma_*(\tilde{\mathcal{F}}(E)) \otimes \mathcal{O}_X(-H)) &\simeq H^i(\tilde{X}, \tilde{\mathcal{F}}(E) \otimes \sigma^* \mathcal{O}_X(-H)) \\ &= H^i(\tilde{X}, \tilde{\mathcal{F}}(-\tilde{H})) \\ &= 0 \end{aligned}$$

which completes the proof. \square

Remark 4. When X is a smooth regular surface, Noma found an equivalence condition for the very ampleness of the analogous line bundle in terms of the position of points for the blowup center [13].

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